Topologically Ordered and Dimerized Phases in Frustrated Spin Ladders

In honor of Igor Dzyaloshinskii

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Grenoble, 9 September 2011

- И.Е. Дзялошинский, *Теория геликоидальных структур в антиферромагнетиках. I. Неметаллы*, ЖЭТФ, 46 (4), 1420-1437 (1964) [I.E. Dzyaloshinskii, *Theory of helicoidal structures in antiferromagnets .1. Nonmetals*, Sov. Phys. JETP 19, 960 (1964)].

- И.Е. Дзялошинский, *Теория геликоидальных структур в антиферромагнетиках. II. Металлы*, ЖЭТФ, 47(1), 336-348 (1964) [I.E. Dzyaloshinskii, *Theory of helicoidal structures in antiferromagnets .2. Metals*, Sov. Phys. JETP 20, 223 (1965)].

- И.Е. Дзялошинский, *Теория геликоидальных структур в антиферромагнетиках. III*, ЖЭТФ, 47(3), 992-1002 (1964) [I.E. Dzyaloshinskii, *Theory of helicoidal structures in antiferromagnets .3*, Sov. Phys. JETP 20, 665 (1965)].

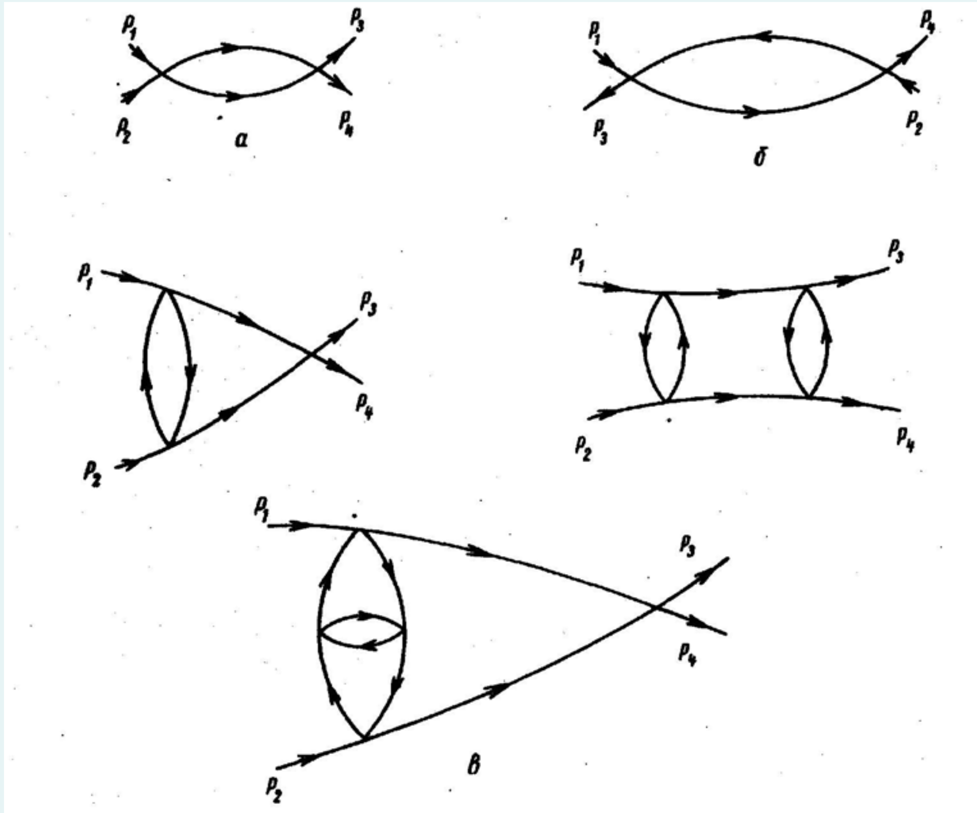
Ю.А. Бычков, Л.П. Горьков, И.Б. Дзялошинский, *Об одномерной сверхпроводимости*, Письма в ЖЭТФ, 2 (3), 146-152 (1965) [Yu.A. Bychkov, L.P. Gor'kov, I.E. Dzyaloshinskii, *Concerning One-dimensional Superconductivity*, JETP Lett., 2 (3), 92-95 (1965)].

ОБ ОДНОМЕРНОЙ СВЕРХПРОВОДИМОСТИ

Ю.А.Бычков, Л.П.Горьков, И.Е.Дзялошинский

Появление работы Литтла [I] вызвало большой интерес к одномерным проводящим системам, каковыми являются молекулы некоторых линейных полимеров. В связи с этим нами была исследована модель

Ю.А. Бычков, Л.П. Горьков, И.Е. Дзялошинский, *О возможности явлений типа сверхпроводимости в одномерной системе*, ЖЭТФ, 50(3), 738-758 (1966) [Yu.A. Bychkov, L.P. Gor'kov, I.E. Dzyaloshinskii, *Possibility of Superconductivity Type Phenomena in a One-Dimensional System*, Sov. Phys. JETP 23(3), 489-501 (1966)].



$$\gamma_{2,3}(\eta,\xi) = s_{2,3}(\eta) + s_{2,3}(\eta)\int_{\eta}^{\xi}\gamma_{2,3}(\varsigma,\xi)\,d\varsigma + \int_{0}^{\eta} s_{2,3}(\varsigma)\gamma_{2,3}(\varsigma,\xi)\,d\varsigma, \quad (\eta<\xi);$$

$$\gamma_{\pm}(\xi,\eta) = \sigma_{\pm}(\xi) \mp \sigma_{\pm}(\xi)\int_{\xi}^{\eta}\gamma_{\pm}(\varsigma,\eta)\,d\varsigma \mp \int_{0}^{\xi}\sigma_{\pm}(\varsigma)\gamma_{\pm}(\varsigma,\eta)\,d\varsigma, \quad (\eta>\xi);$$

$$\gamma_{1}(\xi,\xi) = s_{1}(\xi) + \sigma_{1}(\xi) - \gamma_{1}^{(0)}, \quad \gamma_{2}(\xi,\xi) = s_{2}(\xi) + \sigma_{2}(\xi) - \gamma_{2}^{(0)}, \quad (\eta=\xi).$$

Outline

Ladder models

Haldane and rung-singlet state in simple ladders

Topological order

Magnons vs. spinons

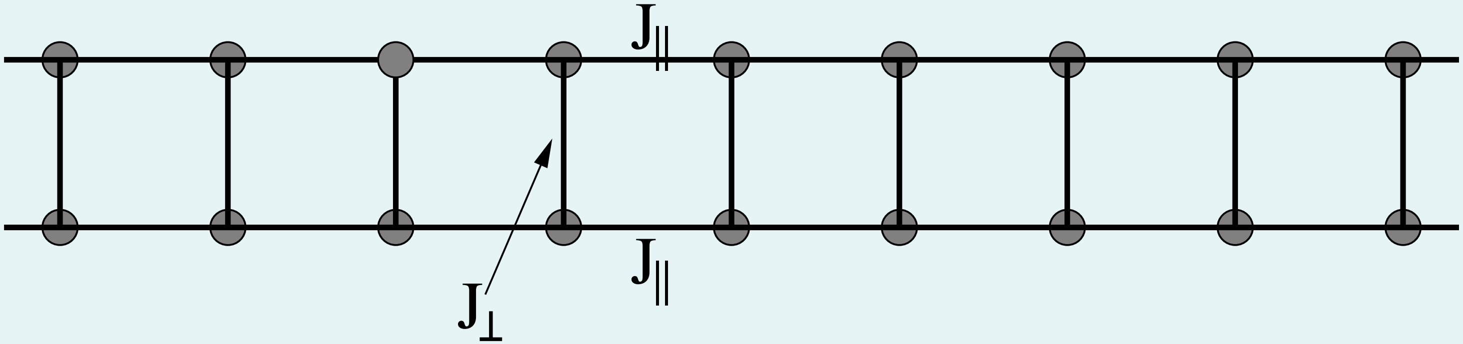
Phase diagram of frustrated ladders:
- zigzag ladder
- diagonal ladder
- cross-coupled ladder

Conclusions

I will consider s= ½ spins sitting on a two-leg ladder, and interacting via an isotropic Heisenberg exchange.

Two basic ladder models:

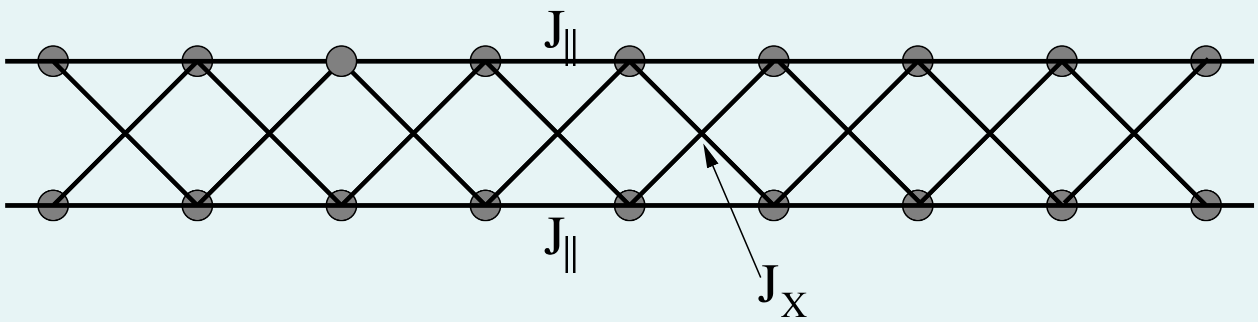
1. Ladder with rung couplings



$$H_0^{(i)} = \sum_l J_\parallel \, \mathbf{S}_l^{(i)} \cdot \mathbf{S}_{l+1}^{(i)} \qquad i = 1, 2$$

$$H_\perp = \sum_l J_\perp \, \mathbf{S}_l^{(1)} \cdot \mathbf{S}_l^{(2)}$$

2. Ladder with diagonal couplings

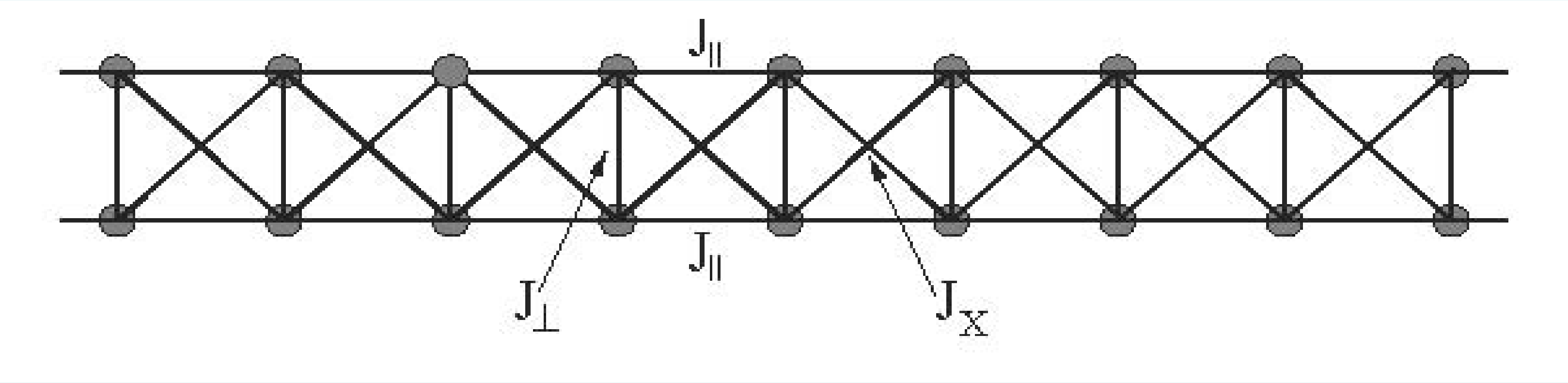


$$H_0^{(i)} = \sum_l J_{\parallel}\, \mathbf{S}_l^{(i)} \cdot \mathbf{S}_{l+1}^{(i)} \qquad i = 1, 2$$

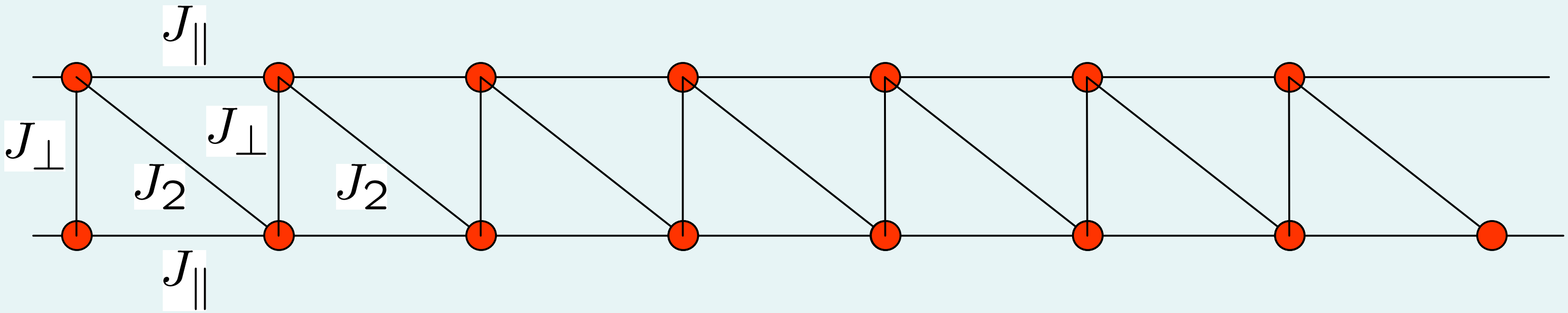
$$H_{\mathrm{X}} = \sum_l J_{\mathrm{X}} \left(\mathbf{S}_l^{(1)} \cdot \mathbf{S}_{l+1}^{(2)} + \mathbf{S}_l^{(2)} \cdot \mathbf{S}_{l+1}^{(1)} \right) \, ,$$

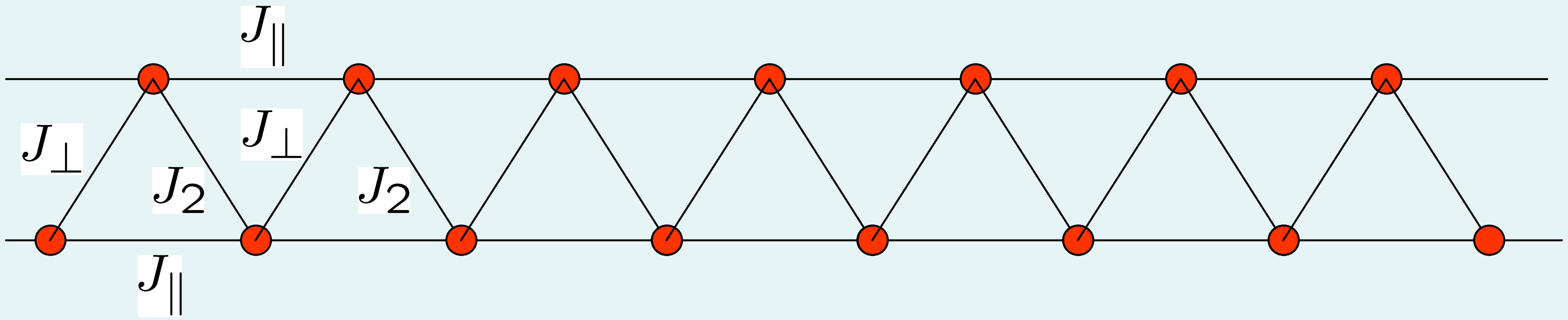
Further models:

3. The cross-coupled ladder has both rung and diagonal couplings

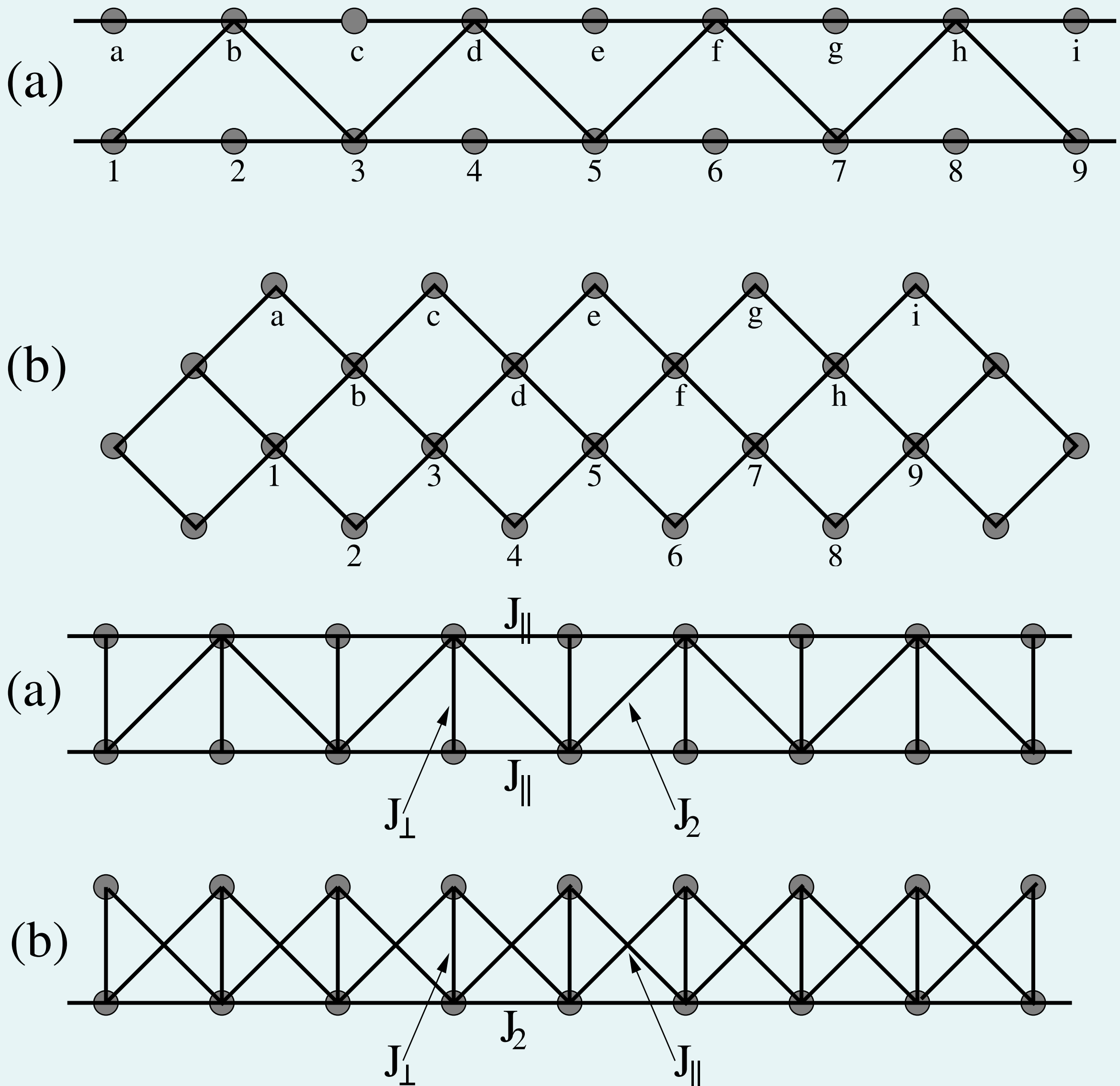


4. If only half of the diagonal couplings are present, this ladder is equivalent to the zigzag ladder



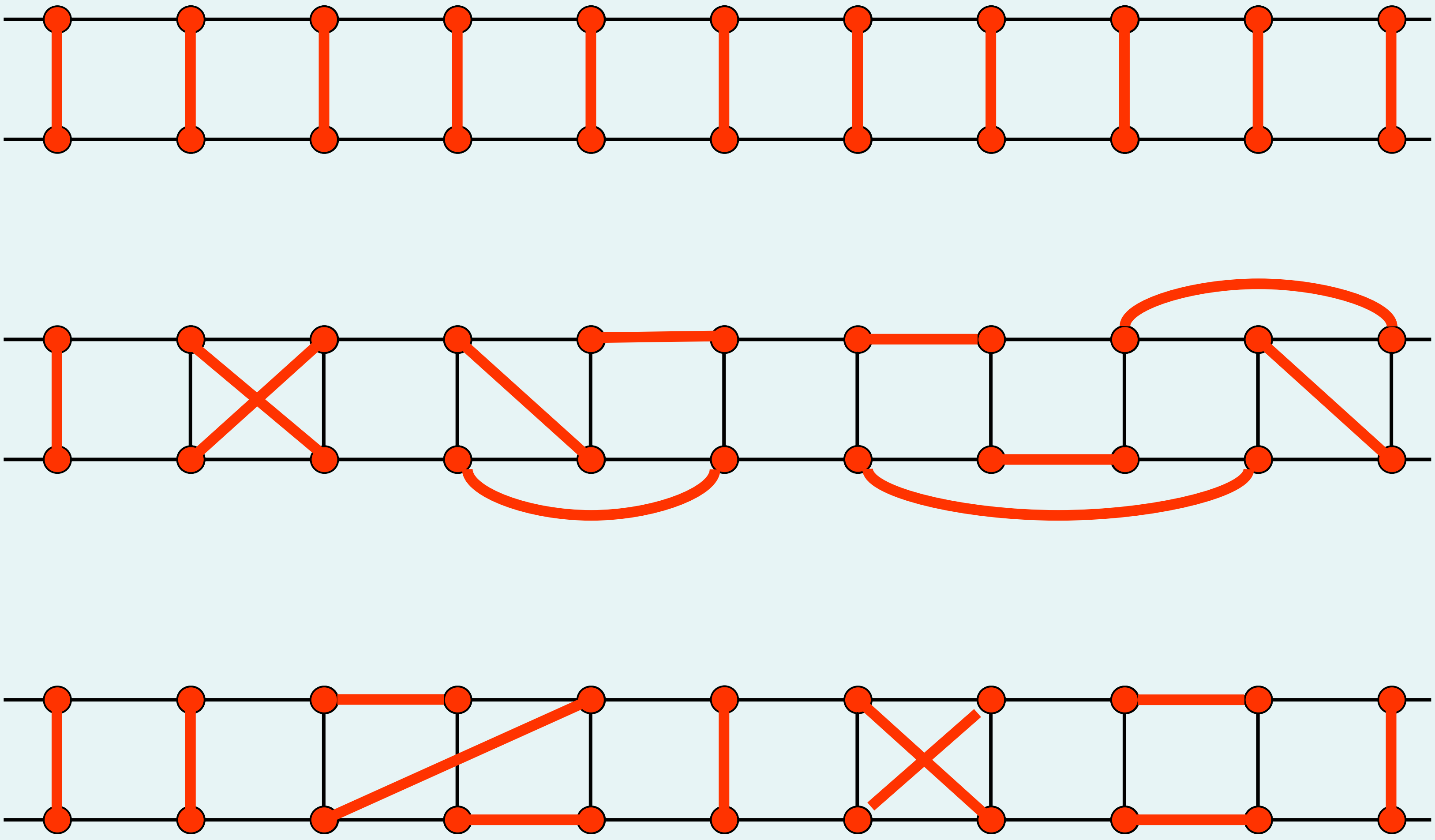


5. We will also consider the „diagonal ladder”



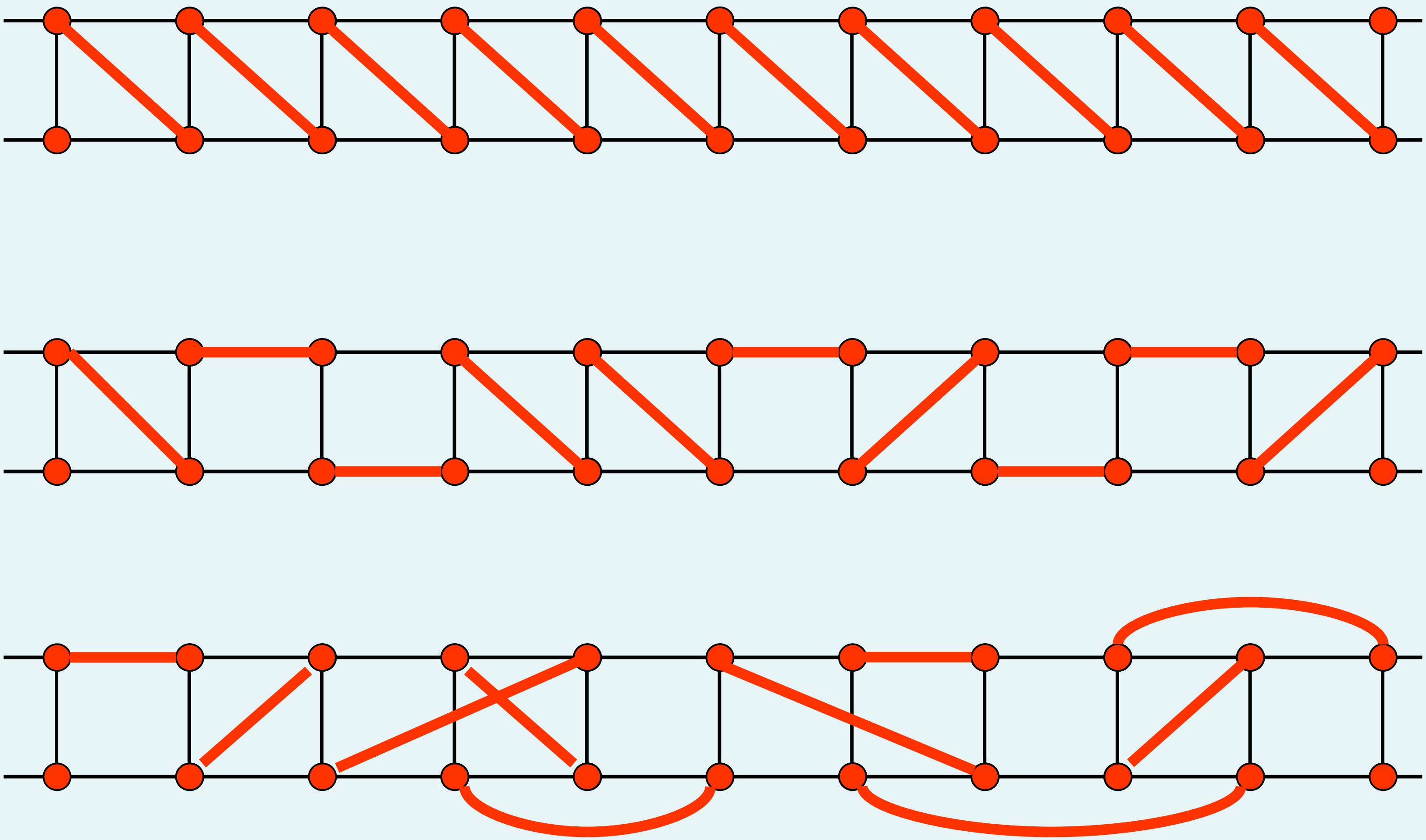
Two types of short-range valence bond states:

1. Rung singlets for antiferromagnetic rung coupling



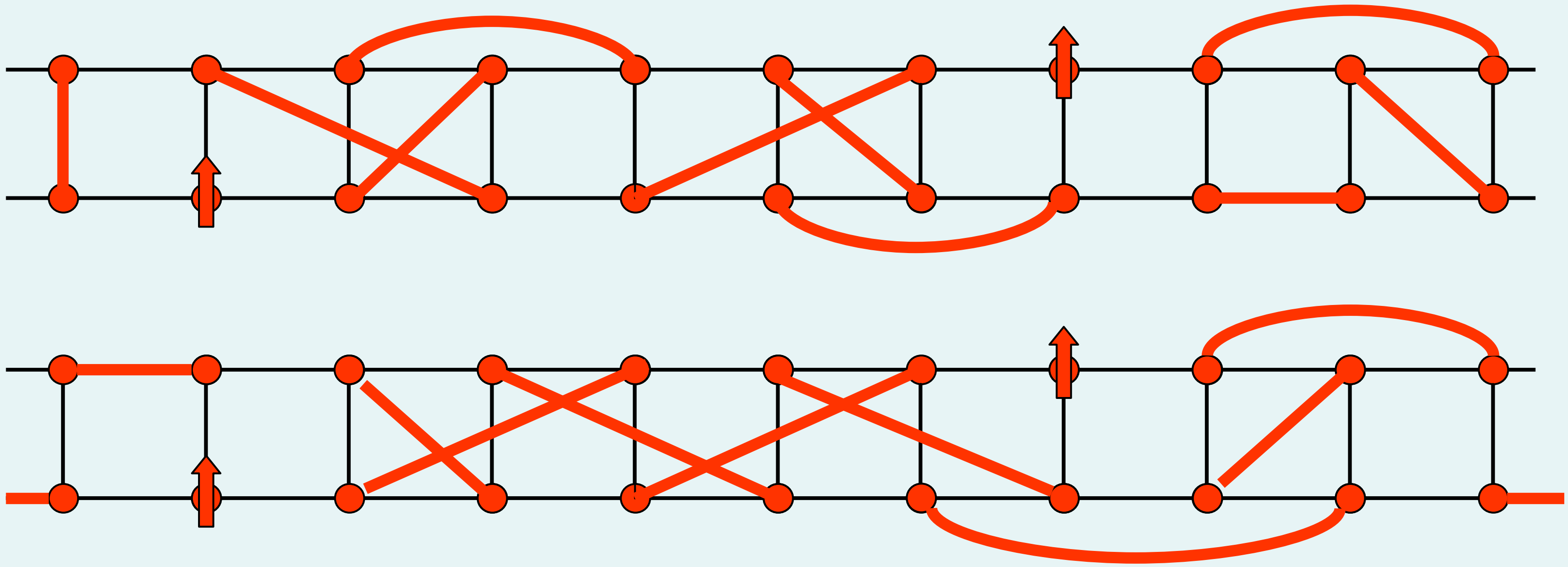
Even-parity ladder

2. Haldane state for ferromagnetic rung coupling or antiferromagnetic diagonal coupling



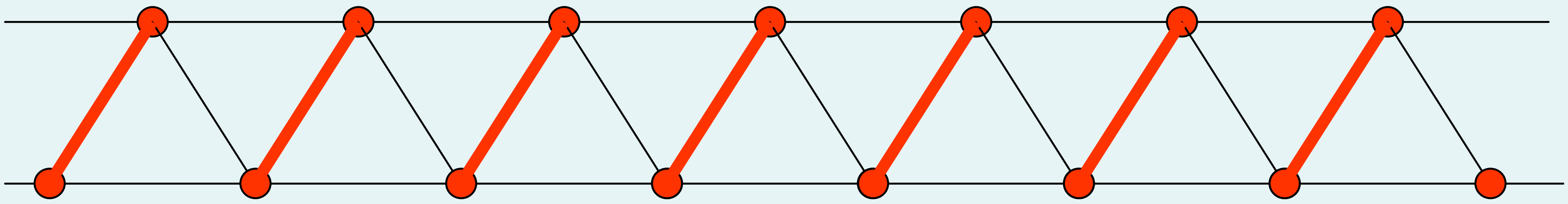
Odd-parity ladder

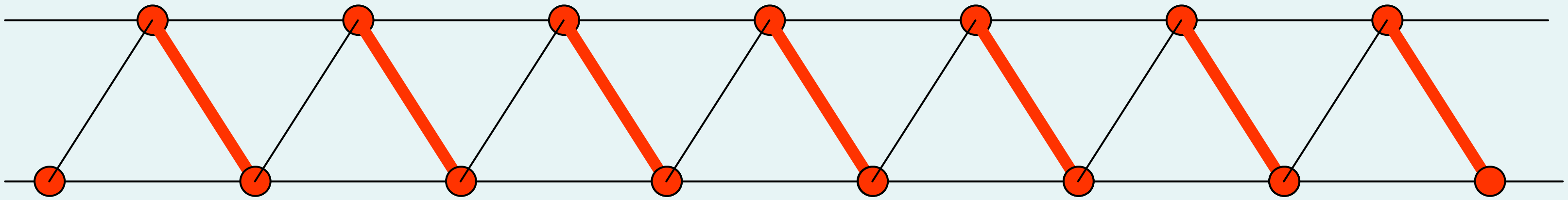
In both cases the elementary excitations are gapped magnons, moving triplets. No deconfined solitons.



The topology of valence bonds, their parity, is different between the two free spins. The energy of these excitations is proportional to the distance between the free spins. This energy confines the spinons into magnons.

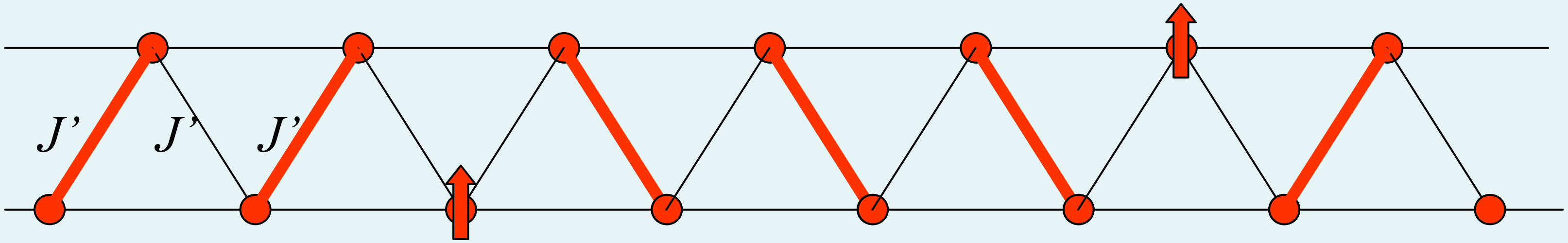
The simplest even- and odd-parity states in the zig-zag ladder are





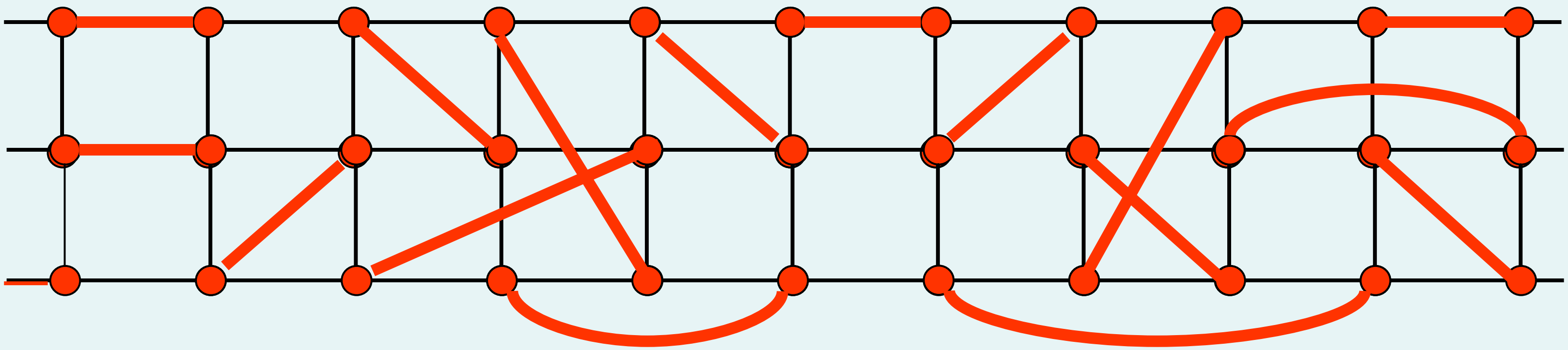
They are exact degenerate ground states if *J’* = *J*/2,

the spinons become deconfined.

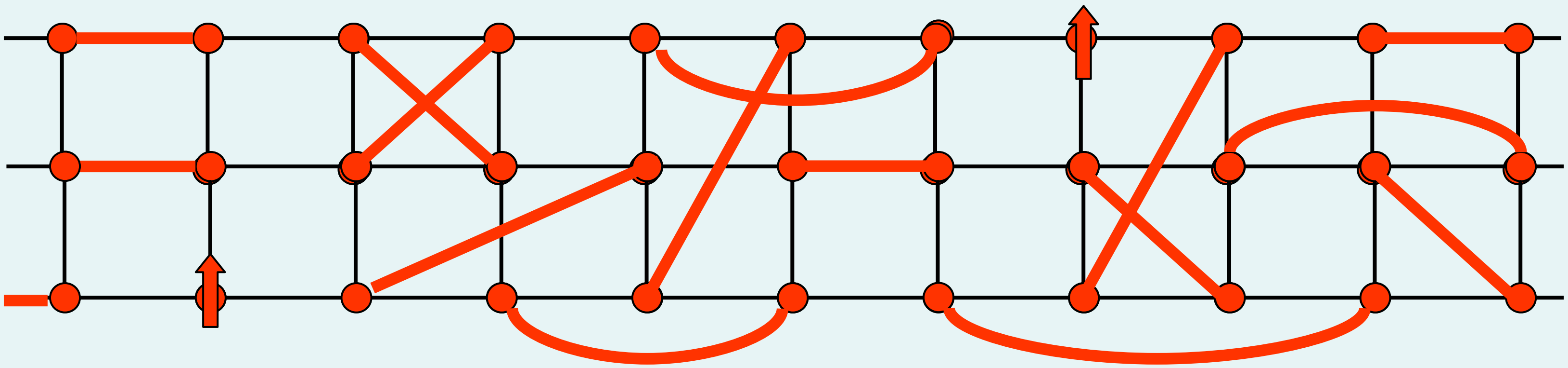


This Majumdar-Ghosh point separates two distinct phases

A three-leg ladder behaves differently.



The number of valence bonds between neighbouring rungs alternates. Spinons can be deconfined.



The basic models are in the rung-singlet or in the Haldane phase for antiferromagnetic couplings.

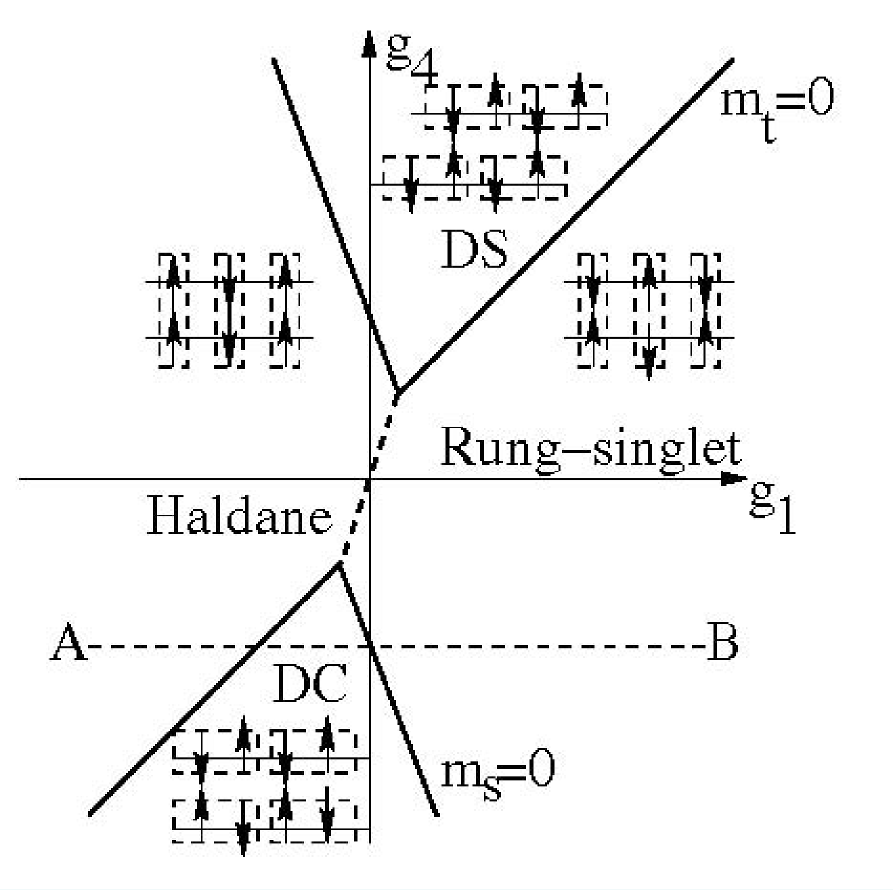
The two types of order can be characterized by two order parameters

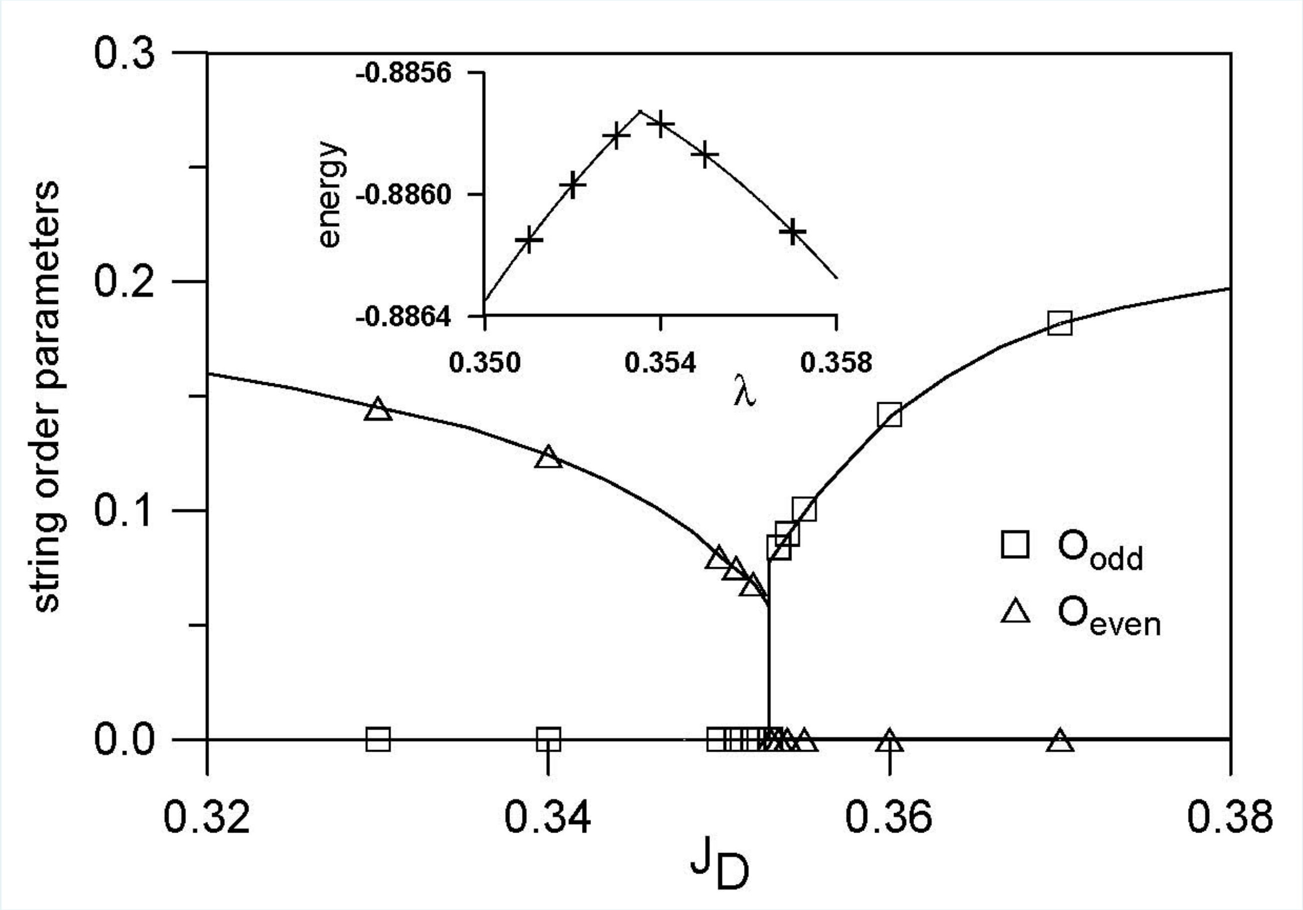
$$\mathcal{O}^{\alpha}_{\rm odd} = - \lim_{|i-j|\to\infty} \left\langle (S^{\alpha}_{i,1} + S^{\alpha}_{i,2}) \ \exp\left(i\pi \sum_{l=i+1}^{j-1} (S^{\alpha}_{l,1} + S^{\alpha}_{l,2}) \right) (S^{\alpha}_{j,1} + S^{\alpha}_{j,2}) \right\rangle ,$$

$$\mathcal{O}^{\alpha}_{\rm even} = - \lim_{|i-j|\to\infty} \left\langle (S^{\alpha}_{i+1,1} + S^{\alpha}_{i,2}) \ \exp\left(i\pi \sum_{l=i+1}^{j-1} (S^{\alpha}_{l+1,1} + S^{\alpha}_{l,2}) \right) (S^{\alpha}_{j+1,1} + S^{\alpha}_{j,2}) \right\rangle$$

What is the type of phase transition between these phases in frustrated models where the rung and diagonal coupling compete? Is there an intermediate phase?

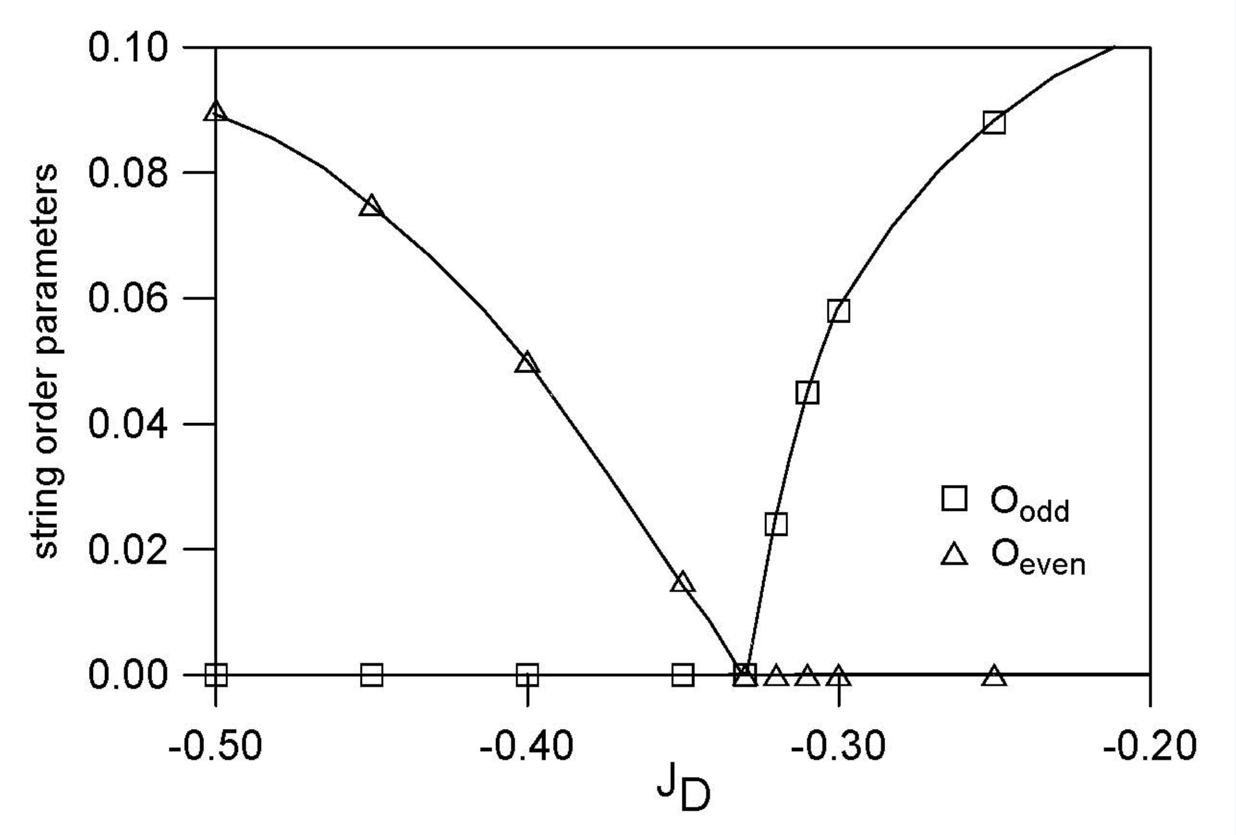
Starykh and Balents: indirect transition via a dimerized state





String order parameter for the cross-coupled ladder for

$$J_\perp + J_X = J_\parallel$$



String order parameter for the cross-coupled ladder for

$$J_\perp + J_\times = -J_\parallel$$

Methods:

Weak-coupling analysis by bosonization and renormalization group techniques,

Numerical studies via the density-matrix renormalization group algorithm

Bosonization

The spin operators are written in terms of the uniform magnetization $J^{\alpha}_{m,\lambda}(x) \quad m = 1,2 \quad \lambda = L, R$
and staggered magnetization $n^{\alpha}_{m}(x) \quad m = 1,2$

$$S^{\alpha}_{i,m} \propto J^{\alpha}_{m,R}(x) + J^{\alpha}_{m,L}(x) + (-1)^{i} n^{\alpha}_{m}(x)$$

Generically the behavior is determined by the term

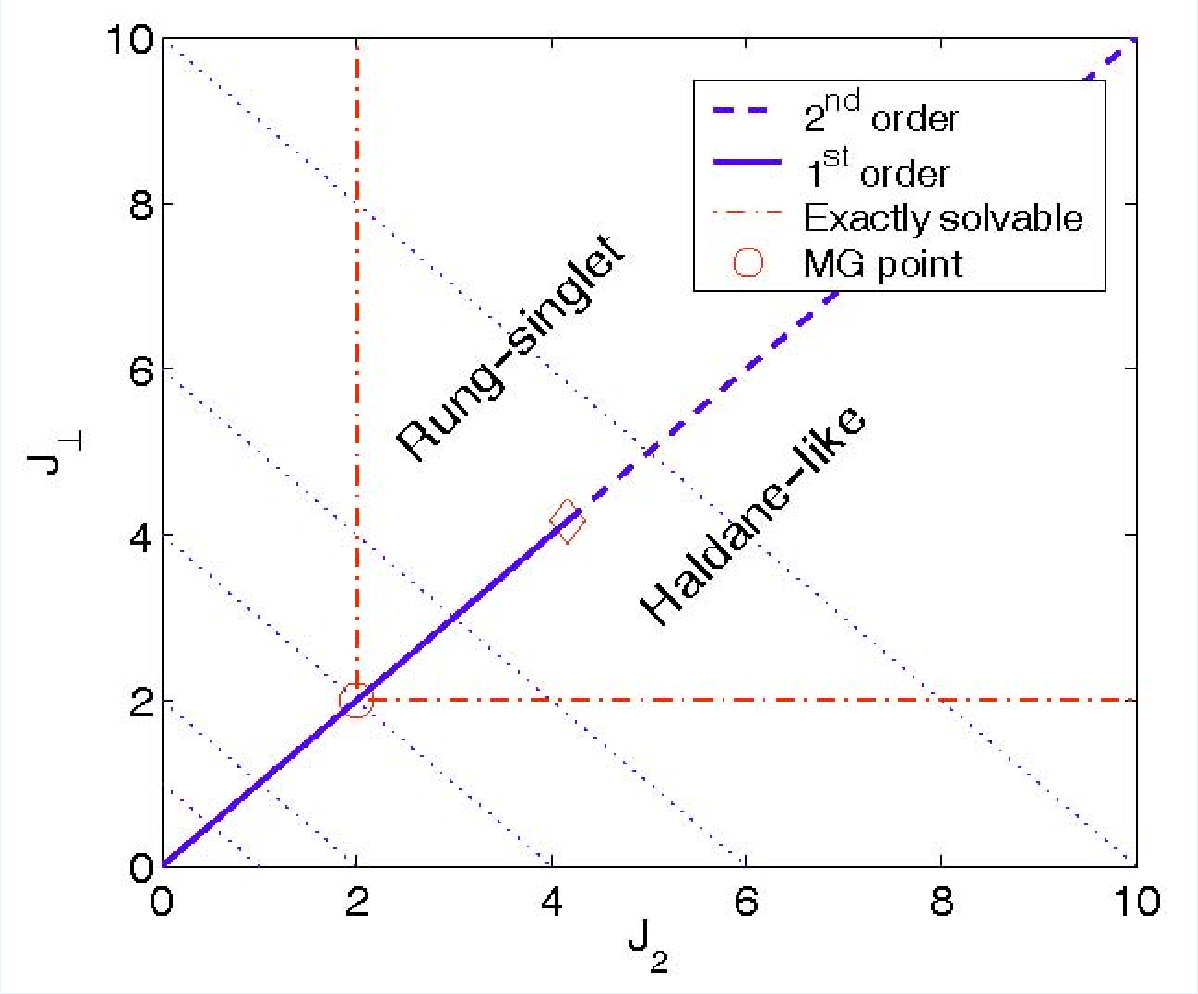
$$n^{\alpha}_{1}(x) \cdot n^{\alpha}_{2}(x)$$

Also the dimerization operator

$$\epsilon_{m}(x) \propto (-1)^{i} S^{\alpha}_{i,m} \cdot S^{\alpha}_{i+1,m}$$

appears in the Hamiltonian

The coefficient of the most relevant term is proportional to $J_\perp - J_2$ for the zigzag ladder. Its sign determines whether we are in the Haldane or in the rung-singlet state



Quantum phase transitions and the DMRG

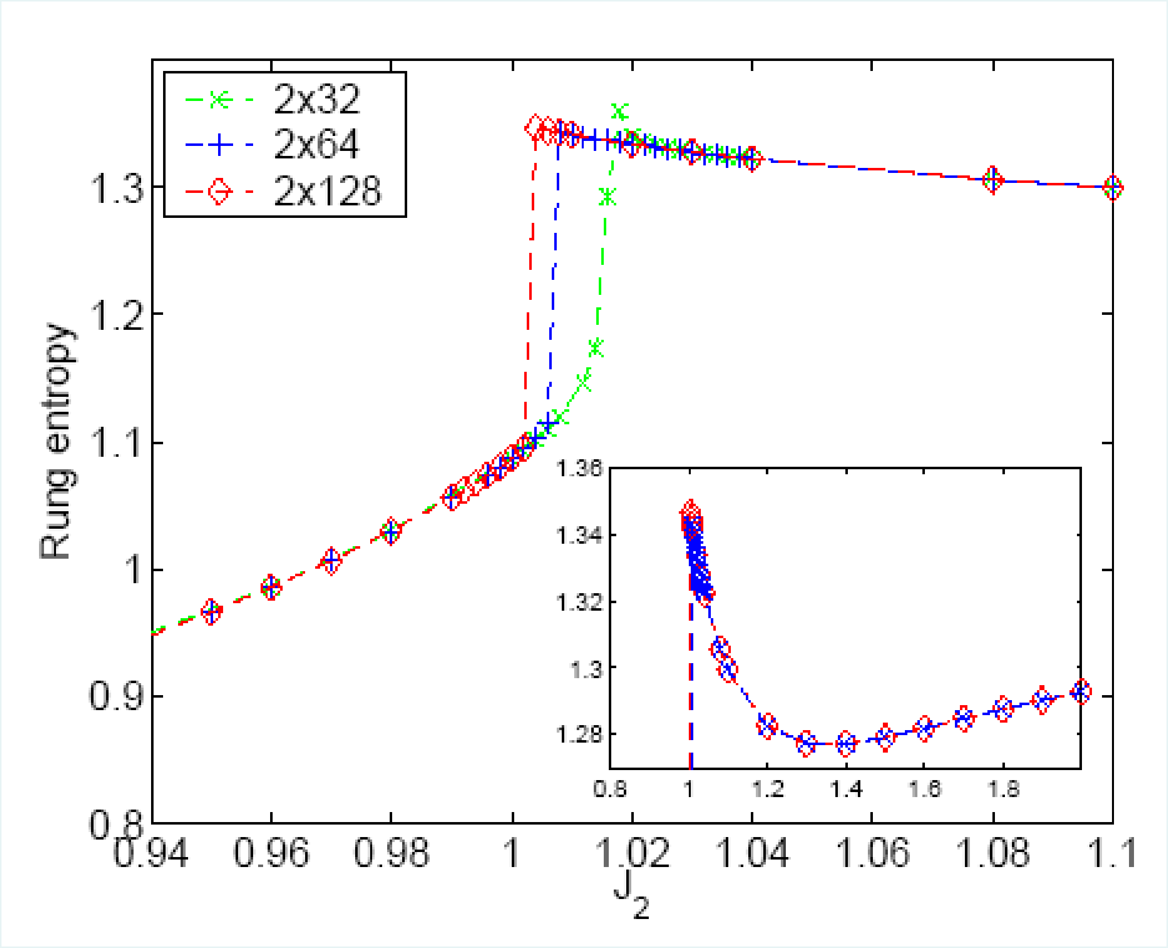
Quantum phase transitions can be conveniently studied
by calculating some measure of entanglement, e.g.,
the entropy of a finite segment of the ladder.

The entropy is readily determined in the DMRG.

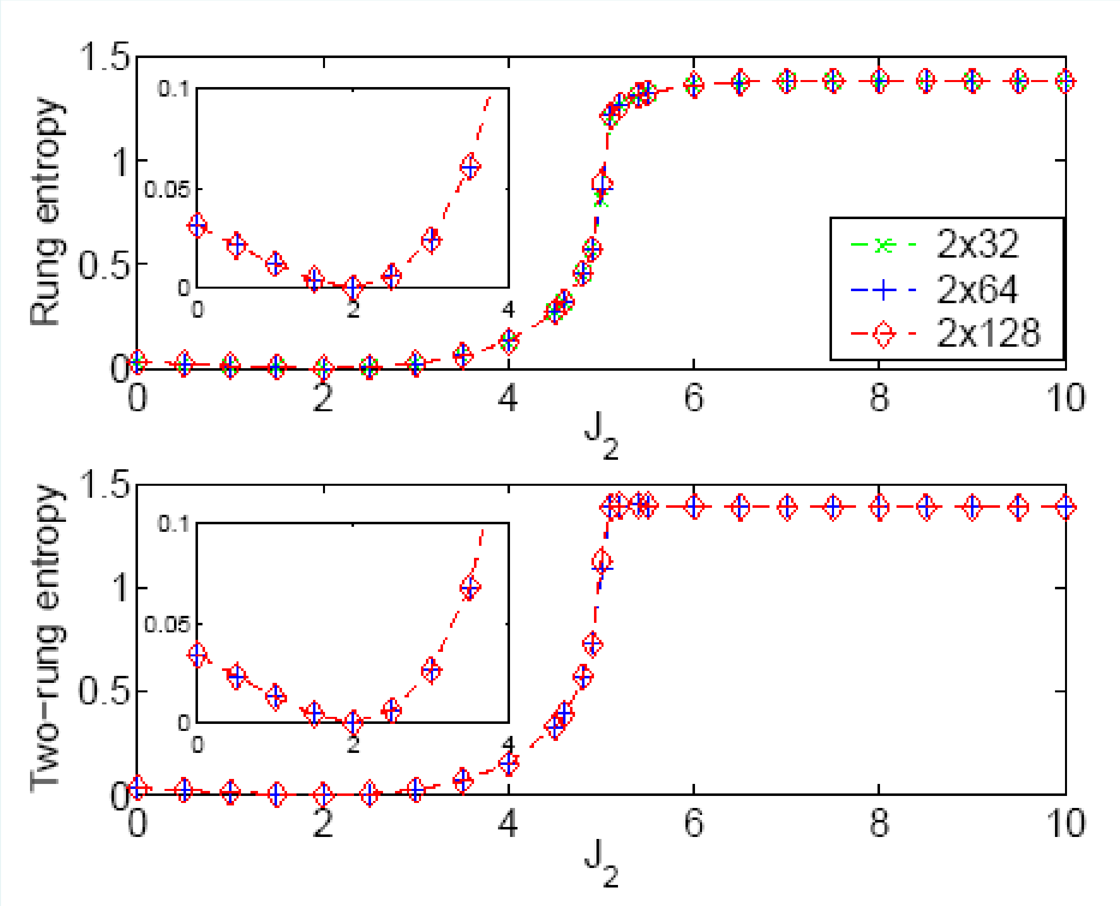
We studied numerical the single-rung and two-rung entropies

$$s_i = -\mathrm{Tr}\, \rho_i \ln \rho_i$$

$$s_{i,i+1} = -\mathrm{Tr}\, \rho_{i,i+1} \ln \rho_{i,i+1}$$



Single-rung entropy of the zigzag ladder for $J_\perp + J_2 = 2J_\|$ for three different lengths



Single-rung and two-rung entropies of the zigzag ladder for $J_\perp + J_2 = 10J_{||}$ for three different lengths

What happens in the diagonal and the cross-coupled ladder?

The coupling constant of the term

$$n_1^\alpha(x) \cdot n_2^\alpha(x)$$

is proportional to

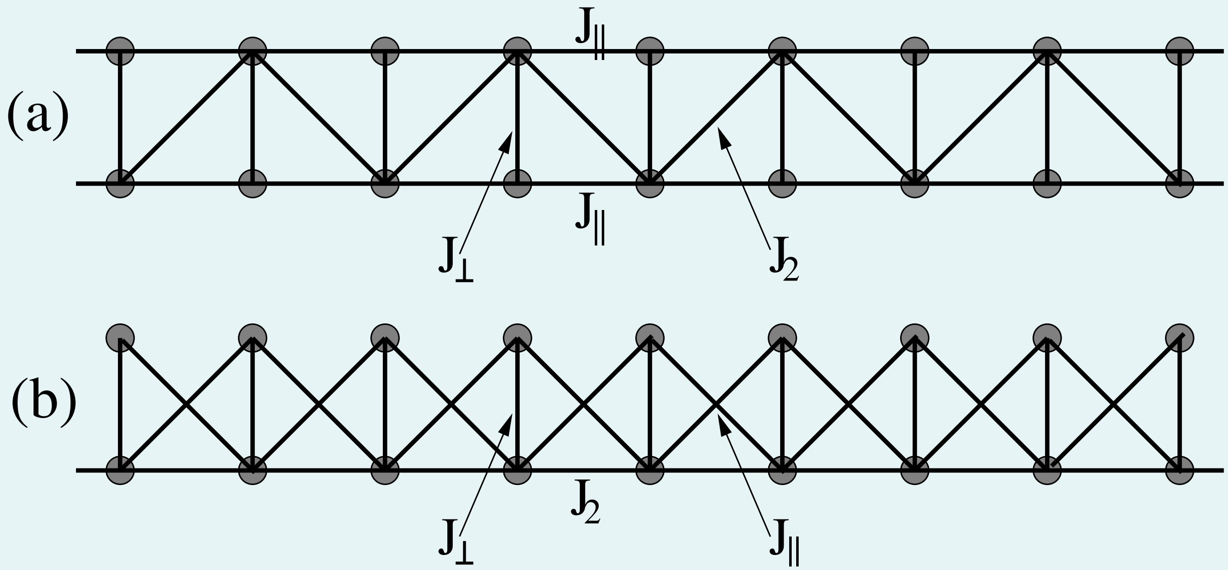
$J_\perp - J_2$	for the diagonal ladder
$J_\perp - 2J_X$	for the cross-coupled ladder.

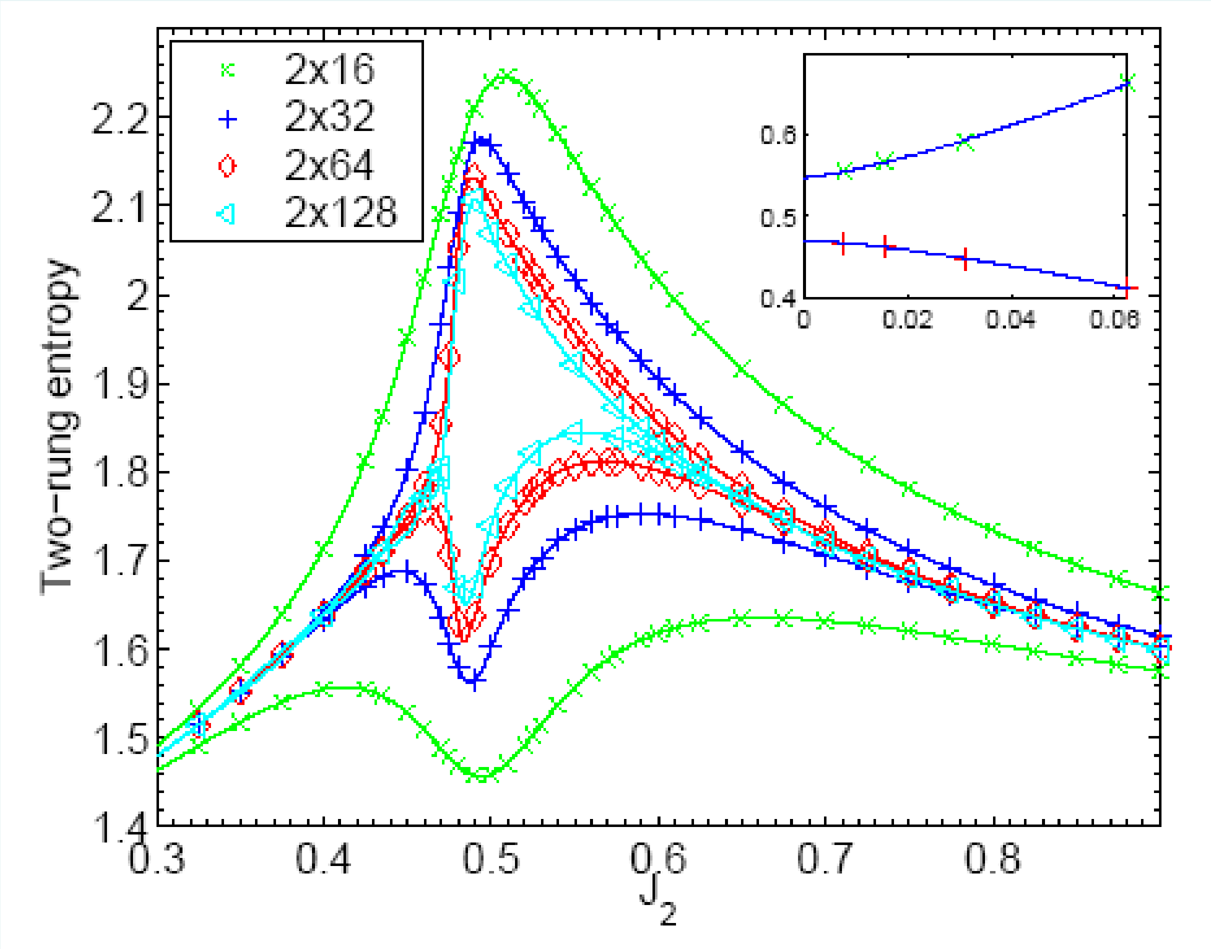
Less relevant operators, like the twist operator

$$n_1^\alpha \partial_x n_2^\alpha(x) - n_2^\alpha \partial_x n_1^\alpha(x)$$

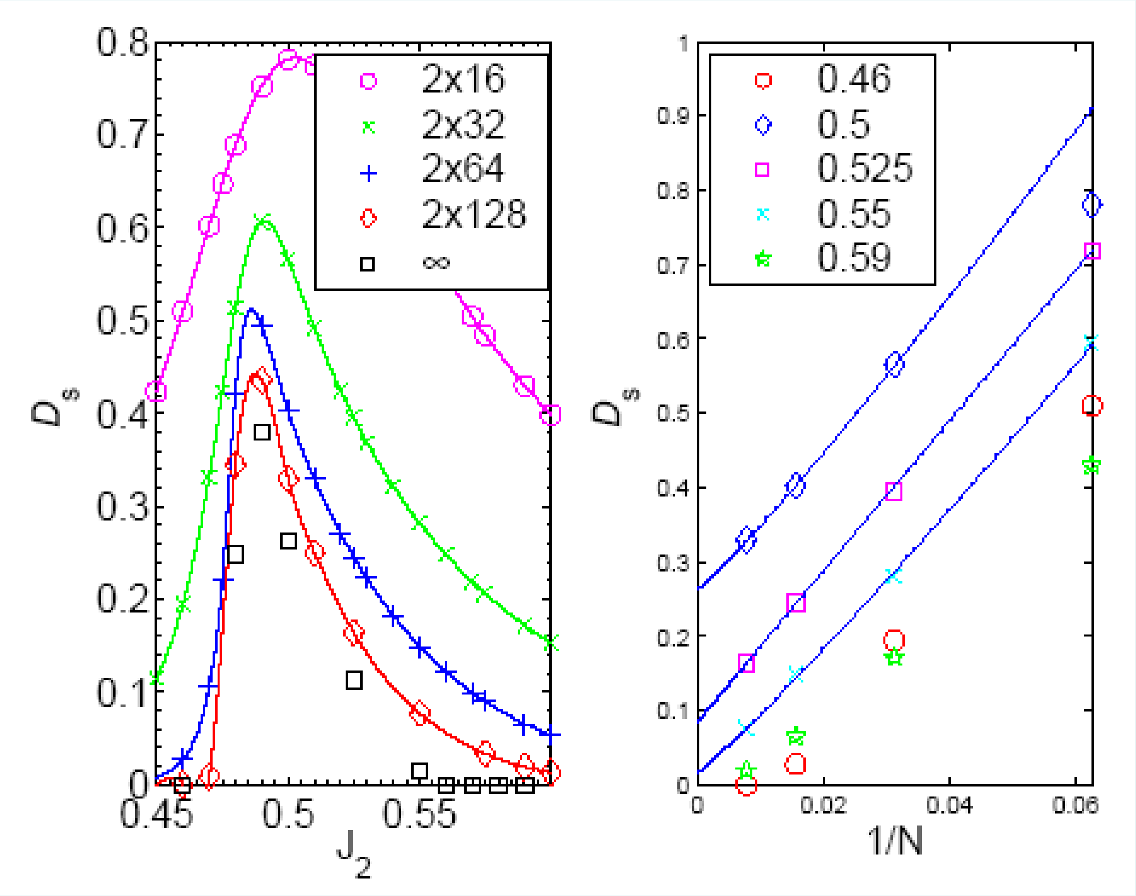
or operators describing dimerized phases may become important along the lines where this coupling vanishes.

Diagonal ladder

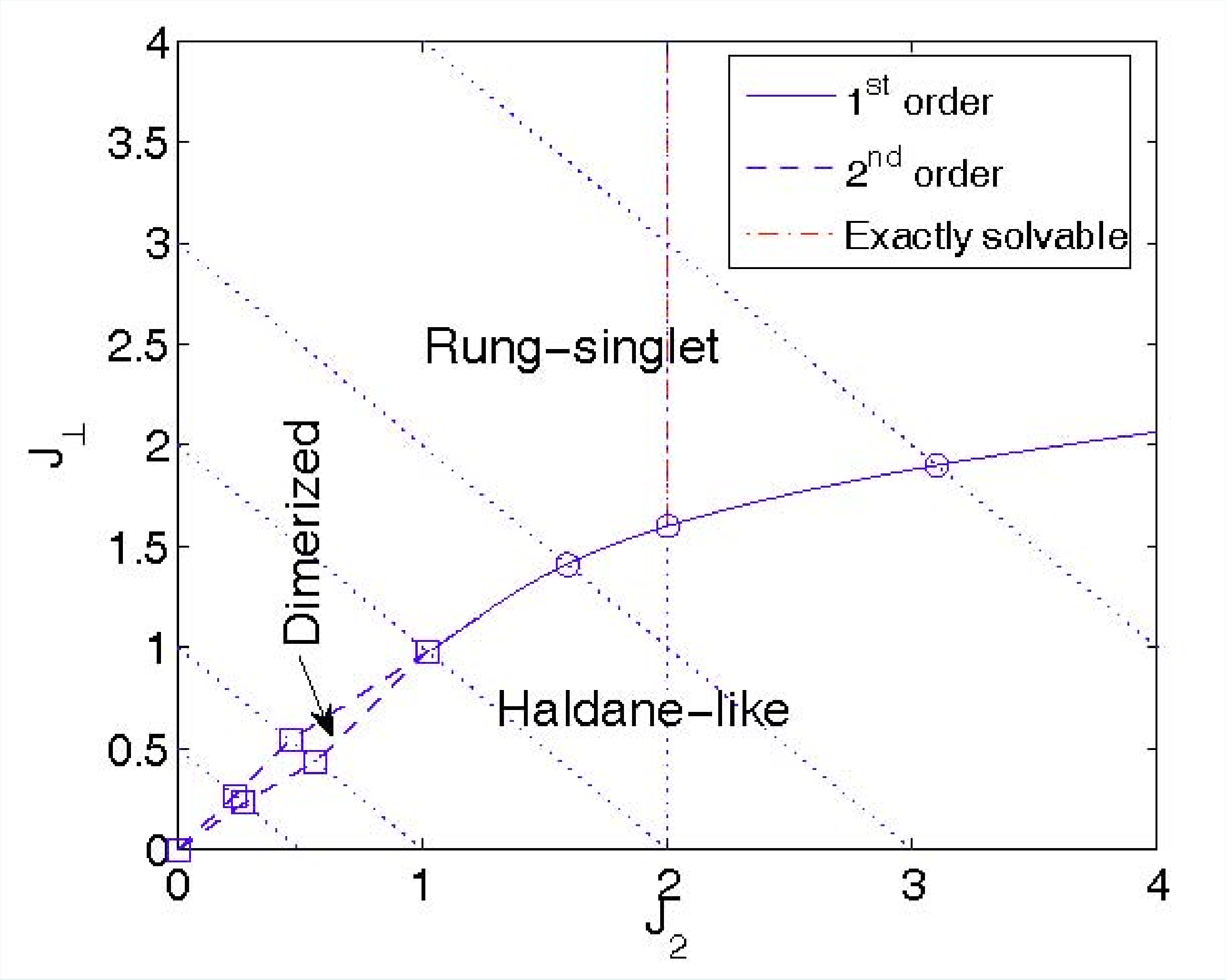




Two-rung entropy of the diagonal ladder for $J_\perp + J_2 = J_\parallel$. The two anomalies do not merge into one for infinitely long ladders

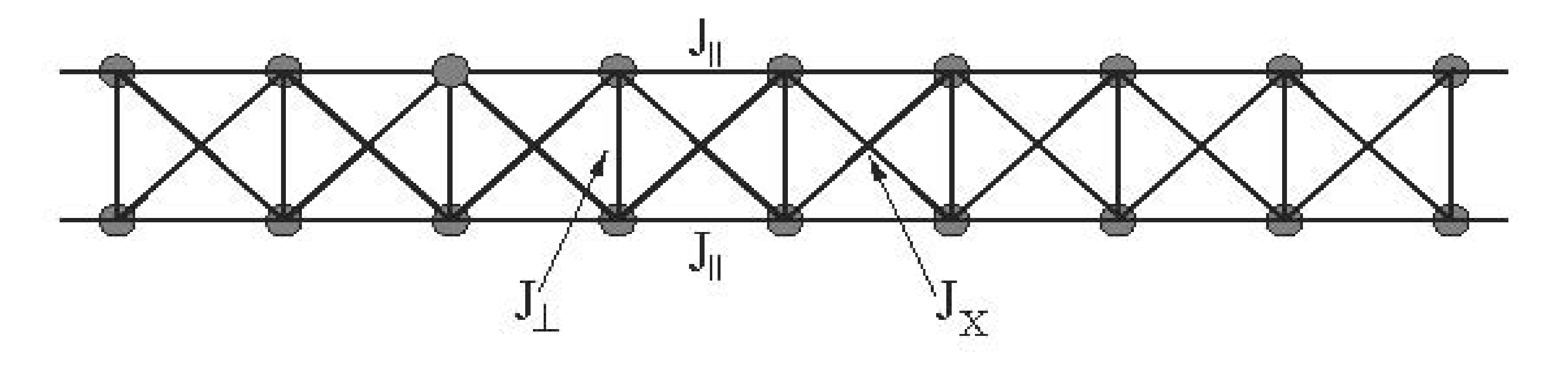


Dimerization entropy, $D_s = s_{l+1,l+2} - s_{l,l+1}$ of the diagonal ladder for $J_\perp + J_\times = J_\|$.



Phase diagram of the diagonal ladder

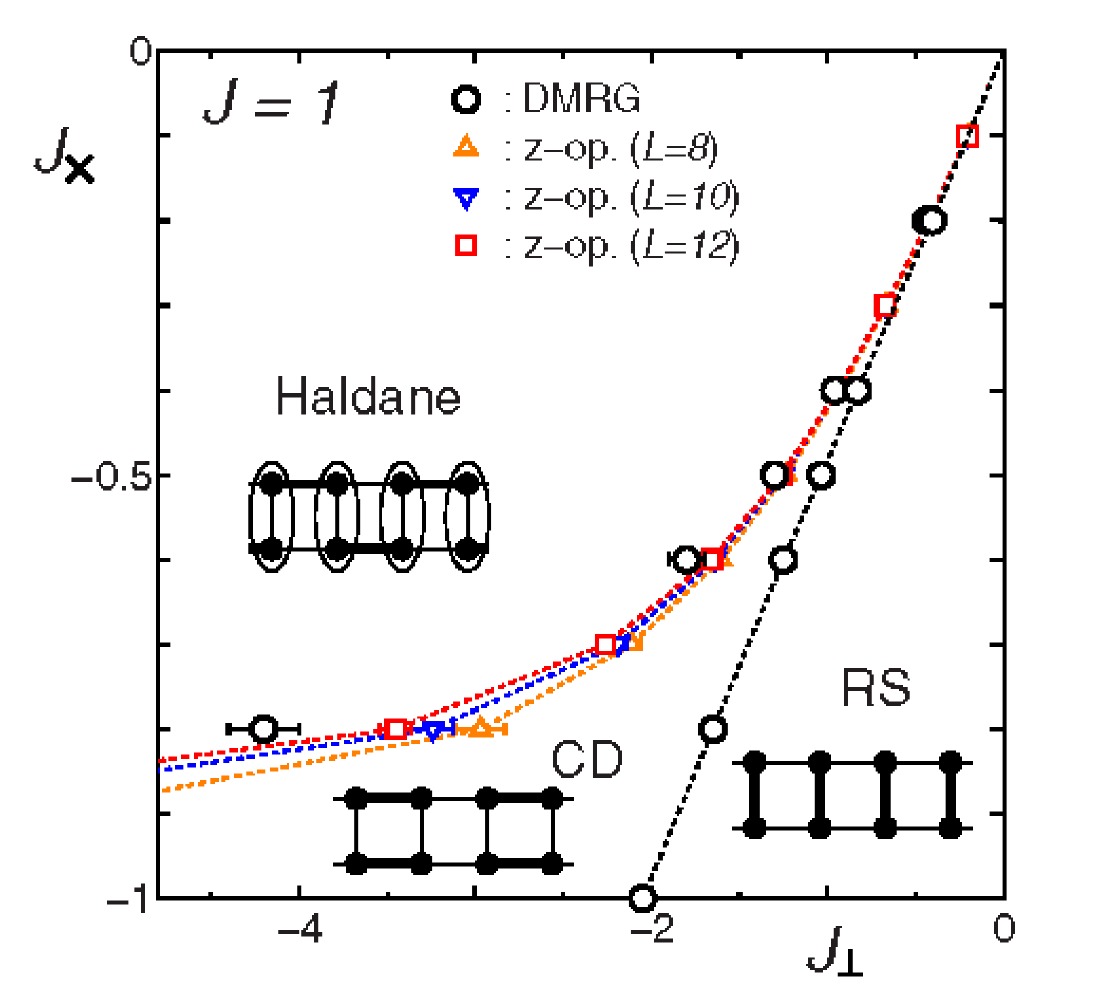
Cross-coupled ladder



The ground state has even parity (rung-singlet state) when the antiferromagnetic rung coupling $J_{\perp}$ is the dominant coupling

The ground state has odd parity (Haldane state) when the cross coupling $J_{\times}$ is the dominant coupling

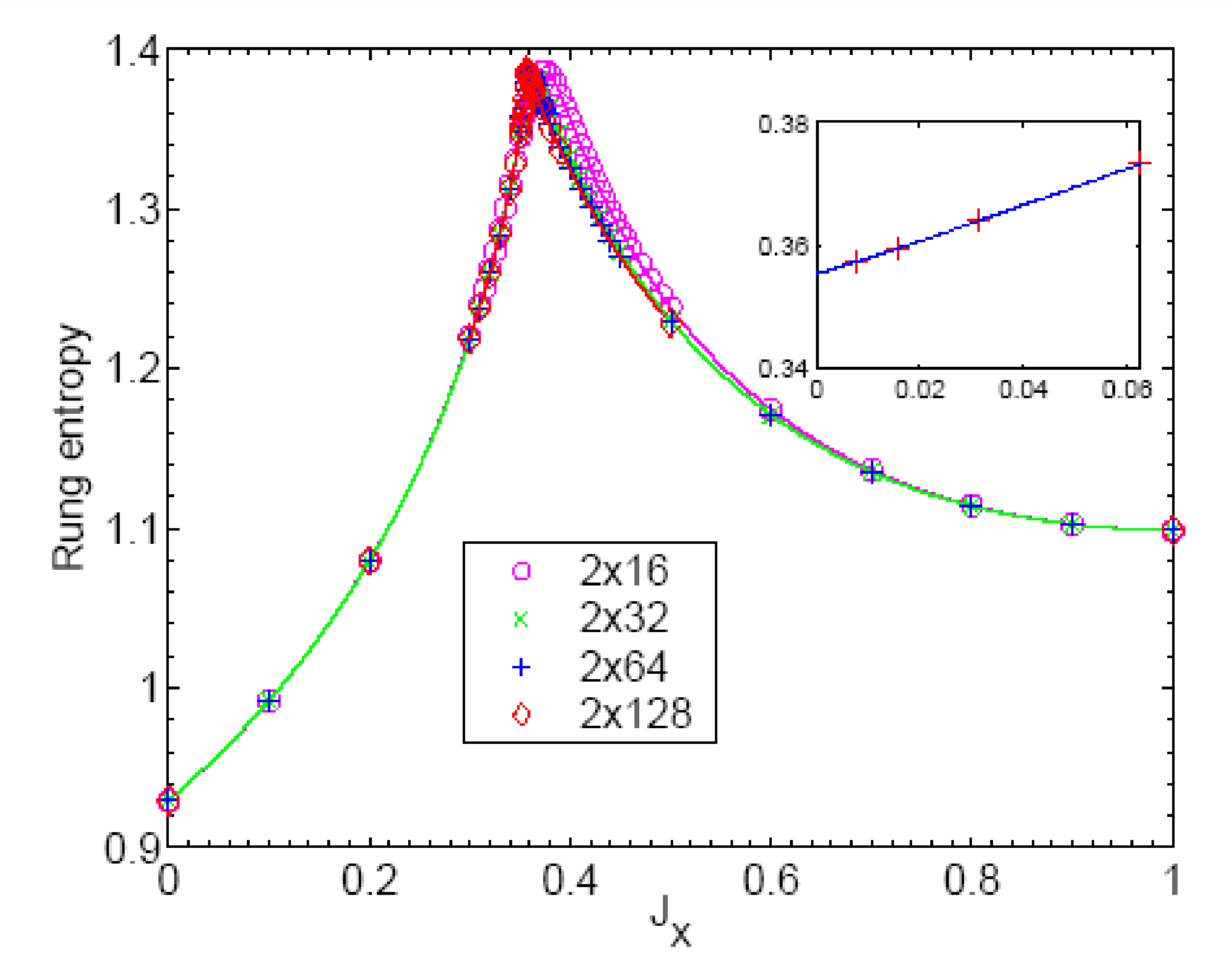
Hikihara and Starykh have found an extended dimerized phase in the cross-coupled ladder when the interleg couplings are ferromagnetic. .



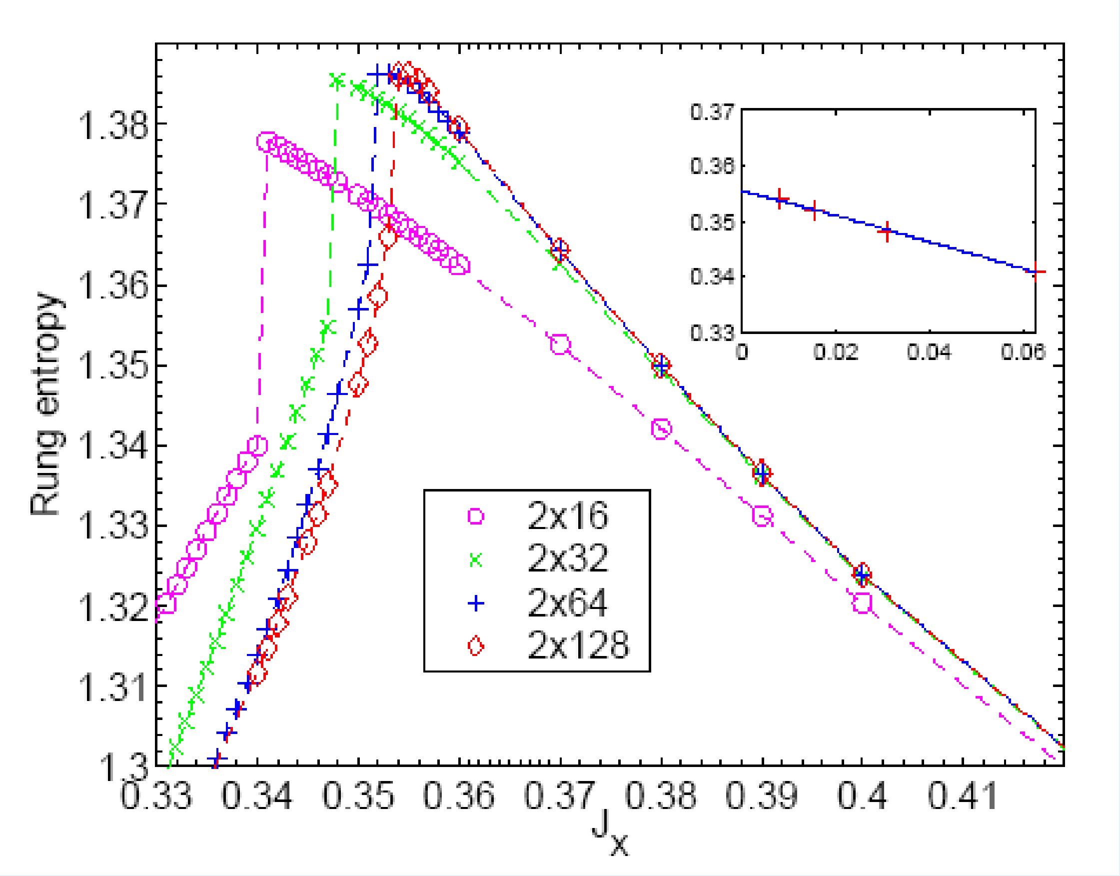
An extended dimerized phase was also found by Vekua and Honecker when second-neighbor intra-leg coupling is switched on.

The numerical work by Hikihara and Starykh indicated a dimerized phase for antiferromagnetic interleg couplings in the cross-coupled ladder, but in an extremely narrow range.

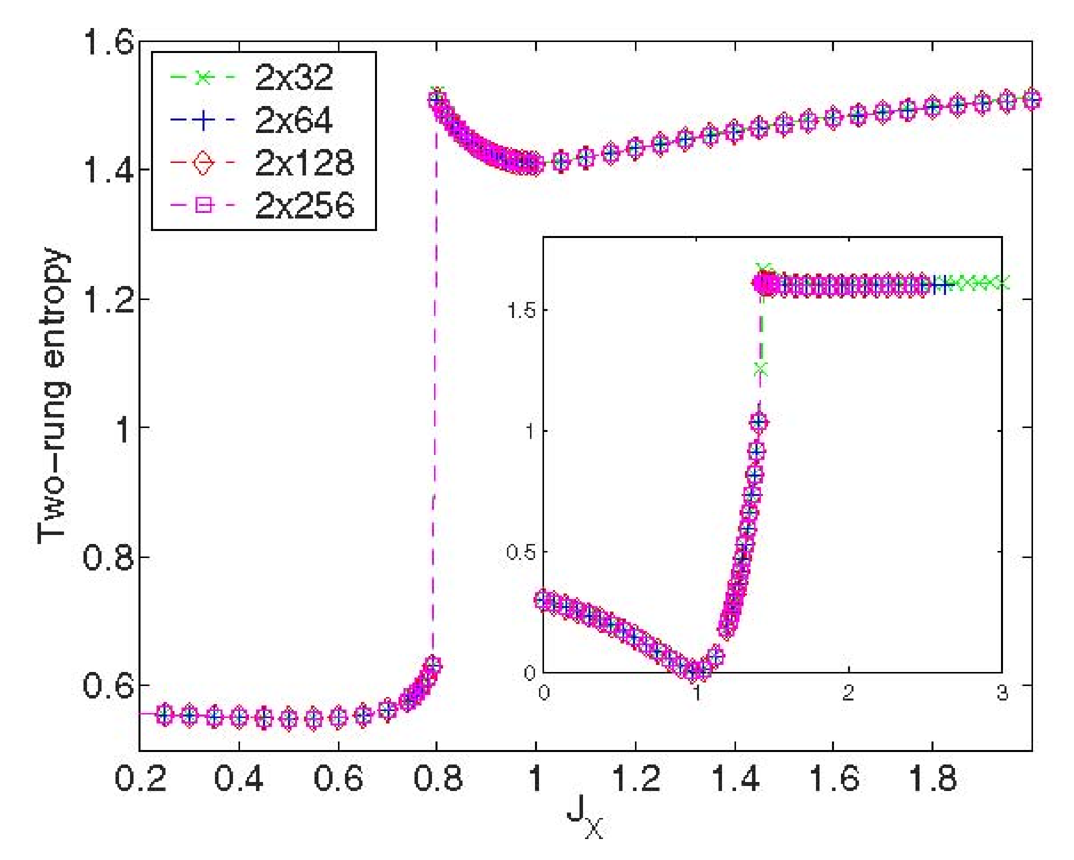
The calculation of the order parameter indicated a direct transition with no intermediate dimerized phase. We used several other methods to check the existence of the dimerized phase for the antiferromagnetic cross-coupled ladder.



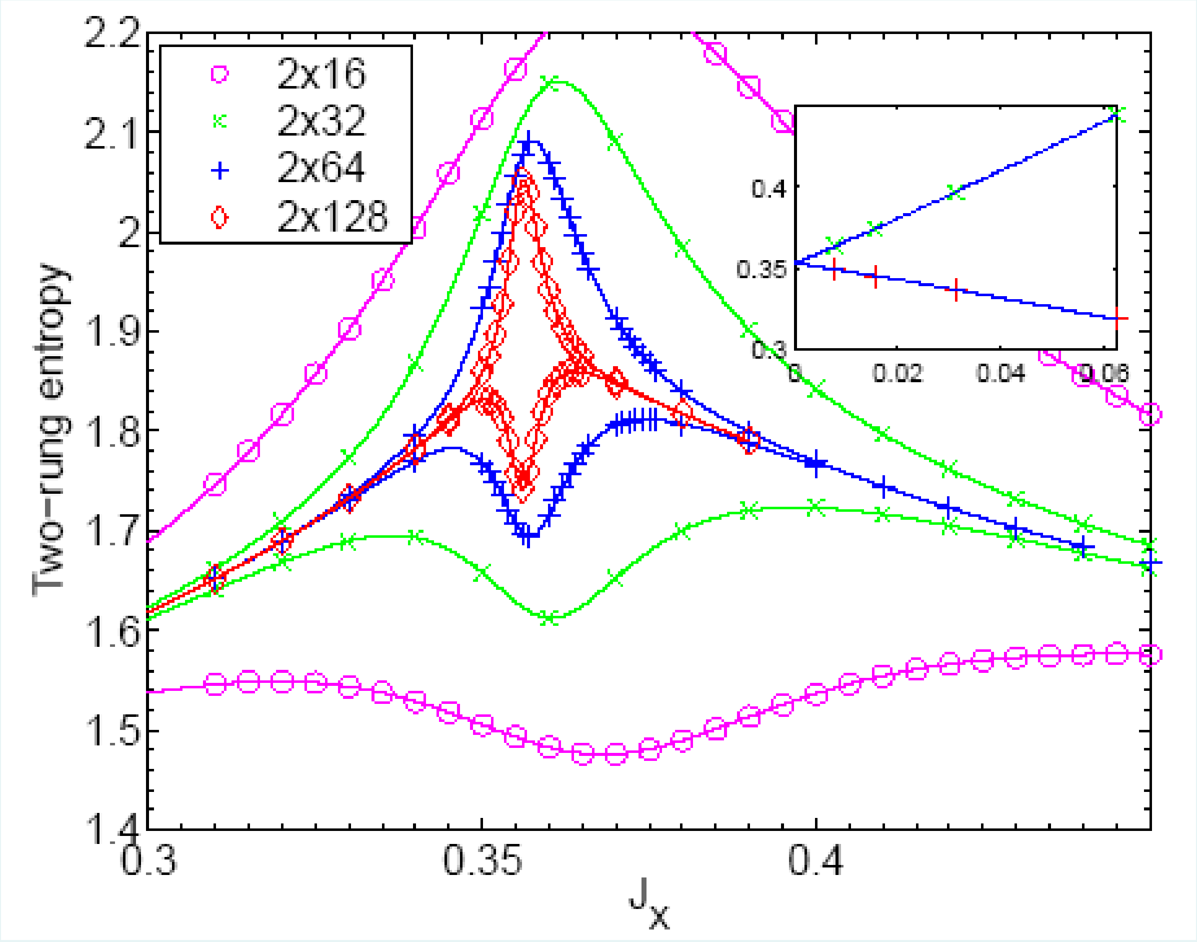
One-rung entropy of the cross-coupled ladder for $J_{\perp} + J_{X} = J_{\parallel}$



One-rung entropy of the cross-coupled ladder for $J_{\perp} + J_{\times} = J_{||}$ when s=1/2 particles are attached to the ends

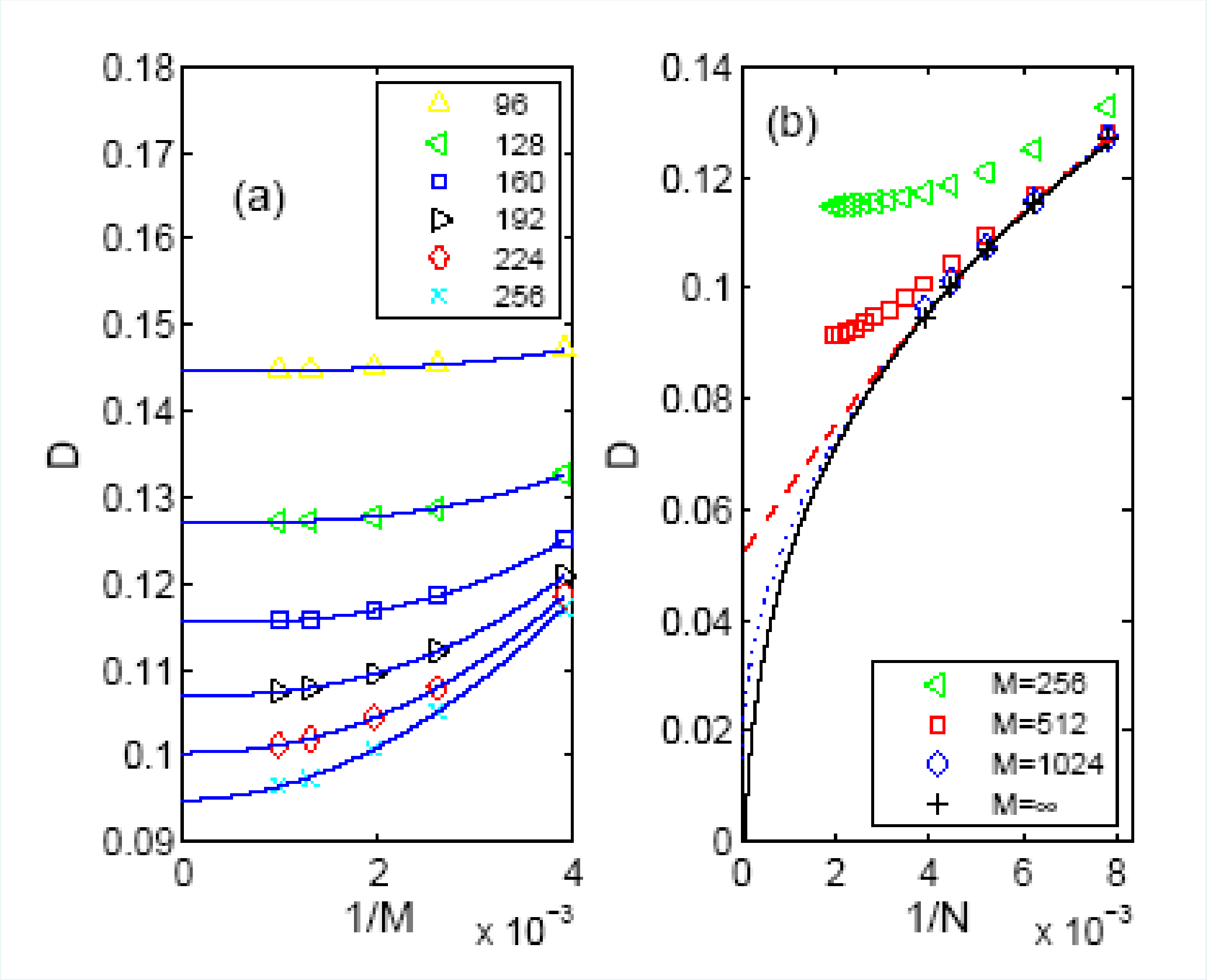


Two-rung entropy of the cross-coupled ladder for $J_\perp + J_X = 2J_\|$ and $J_\perp + J_X = 3J_\|$

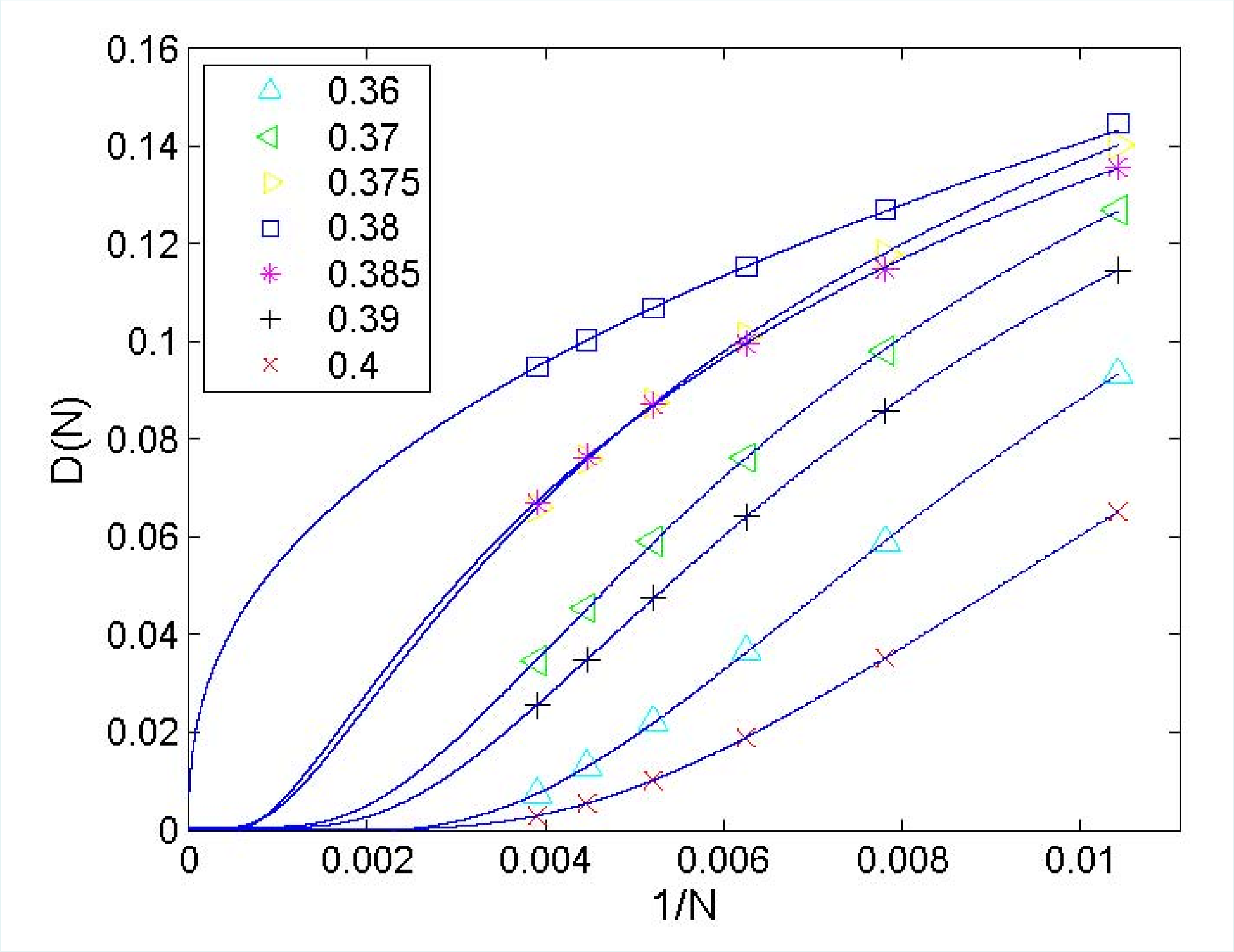


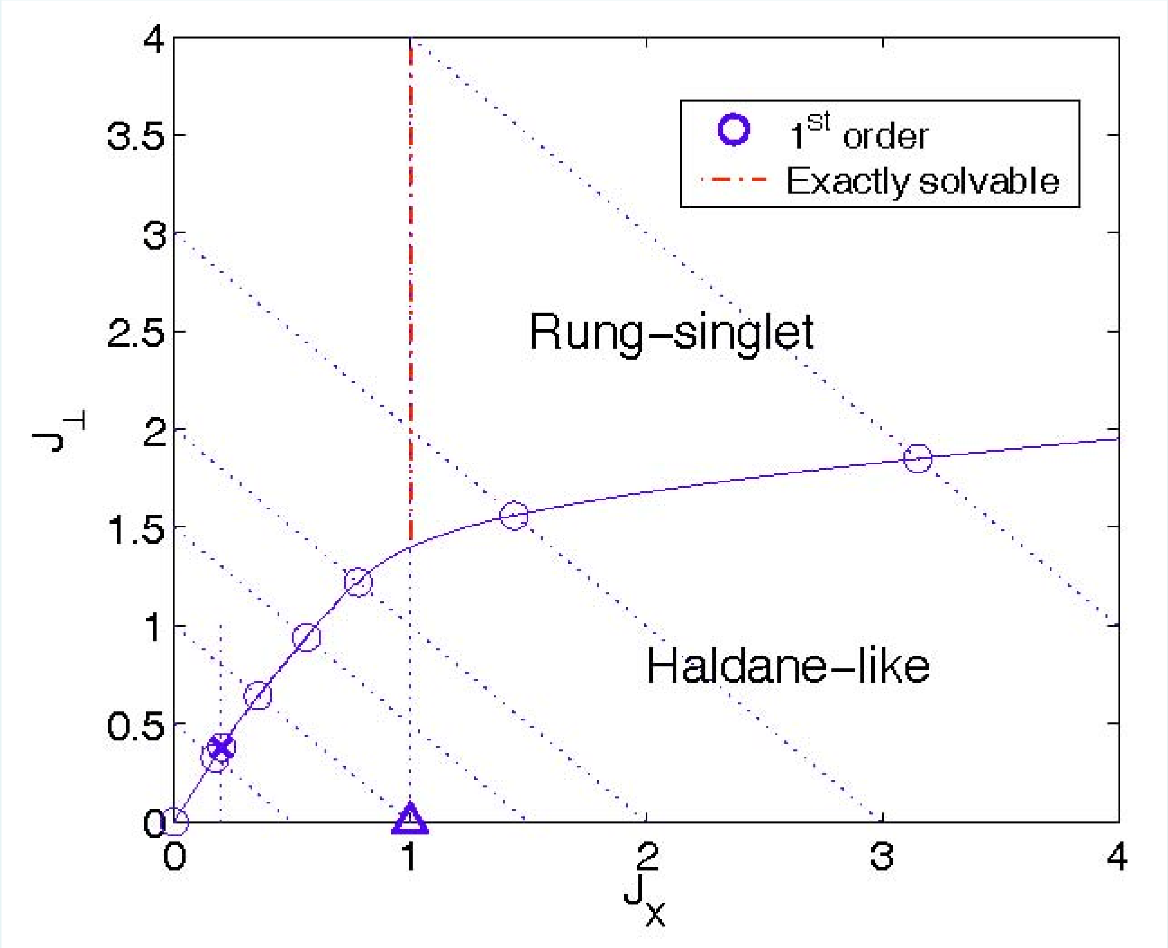
Two-rung entropy of the cross-coupled ladder for $J_{\perp} + J_{X} = J_{\|}$. The two anomalies merge into one for infinitely long ladders

Scaling of the dimer order parameter as a function of the number of kept states in the DMRG procedure and as a function of 1/N



Scaling of the extrapolated dimer order parameter as a function of 1/N near the phase transition point





Phase diagram of the cross-coupled ladder

Conclusions

Spin ladder models typically give rise to an infinite number of operators that could cause dimerization.

No dimerized state in the zigzag ladders.
Direct transition from the rung-singlet to Haldane phase.

Small perturbations may drive the system into the dimerized state.
Such a dimerized phase appears in the diagonal ladder.

The case of cross-coupled ladder is not definitely settled.
The absence of dimerization may be due to a subtle interplay of quantum fluctuations.

Thank you
for your attention